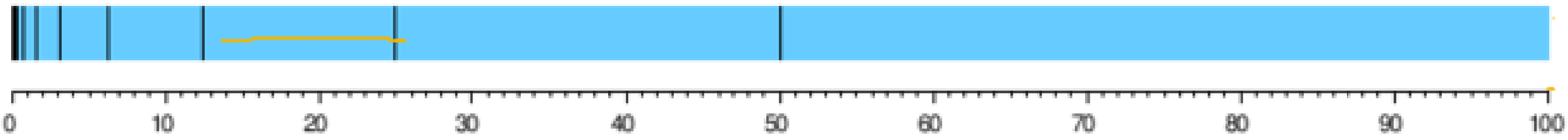
Chapter 2 : Infinite Series & Convergence

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

Motivation (Zeno's paradox) $\quad 2x = x + \frac{x}{2} + \frac{x}{4} + \frac{x}{8} + \cdots$

idea of convergence

$5/8 \quad 5/4 \quad 5/2 \quad 5$



$10 = 5 + \frac{5}{2} + \frac{5}{2^2} + \cdots$

Let $\{u_n\} \to$ seq in $\mathbb{R}$

Then $S_n = u_1 + u_2 + \cdots + u_n$ [another seq of partial sum]

$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$

is called infinite series.

$\lim_{n\to\infty} 5 + \frac{1}{n} = 5$

Theorem : A necessary condition for convergence of an infinite series $\sum u_n$ is $\lim_{n\to\infty} u_n = 0$ (but not sufficient) [ie. $\lim_{n\to\infty} u_n = 0 \not\Rightarrow \sum u_n$ conv]

iff

proof : given $S_n = u_1 + \cdots + u_n$ converges

necessary

sufficient

Let $\lim_{n\to\infty} S_n = l$

$u_n = S_n - S_{n-1} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1})$

$= \lim S_n - \lim S_{n-1} = l - l = 0$

Ex :

(1) $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots$

(2) $1 + 2 + 3 + 4 + \cdots$

(3) $1 - 1 + 1 - 1 + 1 - 1 + \cdots$

4) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \longrightarrow 1/n$

$\left(= \left(\frac{1}{2} - \frac{1}{1}\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \cdots \right)$

nth term $\frac{1}{n(n+1)}$

$= \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)$

$= 1 - 0$

$=$

Examples (whether the following series conv/div)

$\frac{3-1}{1\cdot 2\cdot 3}$

(1) $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots \infty$

(2) $\log 2 + \log\left(3/2\right) + \log\left(4/3\right) + \cdots \infty$

(3) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \cdots \infty \longrightarrow$ diverge

(4) $\sum u_n = 1 + a + a^2 + a^3 + a^4 + \cdots$

$\hookrightarrow$ (will converge for which a?) ✓

$a \in (0,1)$ [fraction]

Theorem [Cauchy's Theorem]

A series $\sum u_n$ converges iff $\forall \varepsilon > 0$, $\exists$ a positive integer N s.t.

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \varepsilon \quad \forall\, n \geq N,\ p \geq 1$$

proof: [Corollary from previous Cauchy's conv Thm]

Ex (1) show that, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is conv

proof: $u_n = \frac{(-1)^{n-1}}{n}$

$$|u_{n+1} + \dots + u_{n+p}| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \dots + \frac{(-1)^{p-1}}{n+p} \right|$$

$$= \left| \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \left(\frac{1}{n+4} - \frac{1}{n+5}\right) + \dots \right| < \frac{1}{n+1} \quad (\downarrow 0 \text{ as } n \to \infty)$$

(H.W.) ?.

H.W.- 1 Show that $\sum \frac{1}{n!}$ is convergent.

[Hint: $n! > 2^{n-1}$]

2 Show that $\sum \frac{1}{n}$ is divergent

Recall [$\left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \right| > \frac{p}{n+p} \geq \frac{m}{m+m} = \frac{1}{2}$]

Will not go to 0

3 Show, $\sum \frac{1}{2n-1}$ is divergent

4 $\sum_{n=1}^{\infty} \cos\left(\frac{x}{n}\right)$ is divergent for all $x \in \mathbb{R}$

[Hint: $\frac{x}{n} \downarrow 0$ as $n \to \infty$
$\Rightarrow \cos\left(\frac{x}{n}\right) \uparrow 1$ as $n \to \infty$]

Theorem: If $\sum u_n$ conv / divg then the following series conv / divg respectively

(1) $c \sum u_n$ $\quad [c \in \mathbb{R}]$

(2) $\sum u_n \pm \alpha$ $\quad$ [for any finite $\alpha \in \mathbb{R}$]

(3) $\sum_{n=p}^{\infty} u_n$ $\quad$ [for $p > 0$, $p \in \mathbb{N}$]

Theorem: $\sum u_n$, $\sum v_n$ convergent

$\Rightarrow \sum (u_n \pm v_n)$ convergent

proof : $\sum u_n$ conv $\Rightarrow$ $\forall\, n \geq m_1$ & $p \geq 1$

$$\left|\sum u_{n+i}\right| = |u_{n+1} + \cdots + u_{n+p}| < \frac{\varepsilon}{2}$$

$\sum v_n$ conv $\Rightarrow$ $\forall\, n \geq m_2$ & $p \geq 1$

$$\left|\sum v_{n+i}\right| = |v_{n+1} + \cdots + v_{n+p}| < \frac{\varepsilon}{2}$$

$$\Rightarrow \left|\sum_{i=1}^{p} (u_{n+i} + v_{n+i})\right|$$

$$\leq \left|\sum_{i=1}^{p} u_{n+i}\right| + \left|\sum_{i=1}^{p} v_{n+i}\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq m \;\&\; p \geq 1$$

where $m = \max\{m_1, m_2\}$

[Proved by Cauchy's Thm]

Corollary: If $\sum u_n$ is conv, then the series $\sum v_n$ (obtained by grouping the terms in the bracket without altering the order of the terms) is also conv.

H.W: (The converse is false)

$$u_n = (-1)^{n-1} = +1 - 1 + 1 - 1 + \cdots$$

$$\overset{?}{=} +1 (-1+1) + (-1+1)$$

$$\overset{?}{=} (+1-1) + (+1-1)$$

$$\overset{?}{=}$$

$\times$ 1 0

Series of positive Terms

$\sum u_n$ is called series of positive terms if $u_n > 0 \ \forall \ n \geq N$

Theorem: $\sum u_n$ be a series of positive terms Then $\sum u_n$ conv iff $S_n = \sum_{i=1}^{n} u_i$ is bdd above.

proof: $S_{n+1} - S_n = u_{n+1} > 0$

$\Rightarrow \{S_n\}$ monotonically increasing

(Recall) $\{S_n\}$ conv iff it is bdd above. (proved)

Ex: If $\sum u_n$ be a conv series of positive reals then show that $\sum u_{2n}$ is conv.

Ans: $\sum u_n$ conv $\Rightarrow \{S_n\}$ bdd above

$\{S_n\}$ → partial sum

$\Rightarrow \exists M \in \mathbb{R}$ s.t

$u_1 + u_2 + \cdots\cdots + u_{2n} \leq M$

$\Rightarrow u_2 + u_4 + \cdots\cdots + u_{2n} \leq M - (u_1 + u_3 + \cdots + u_{2n-1})$

$\Rightarrow u_2 + u_4 + \cdots\cdots + u_{2n} \leq M$

$\Rightarrow \{S'_n\} = \{u_2 + u_4 + \cdots + u_{2n}\}$ bdd above & positive real series

$\Rightarrow \sum u_{2n}$ Conv. (proved)

Ex : $\sum u_n$ be a conv series of +ve $\mathbb{R}$. Show that $\sum u_n^2$ is conv.

Ans : $\sum u_n$ conv $\Rightarrow \{S_n\}$ bdd above

$\Rightarrow \exists M \in \mathbb{R}$ s.t. $u_1 + u_2 + \cdots + u_n \leq M$

$$\Rightarrow (u_1 + u_2 + \cdots + u_n)^2 \leq M^2$$

$$\Rightarrow \underbrace{u_1^2 + u_2^2 + \cdots + u_n^2}_{} \leq M^2 - 2(u_1u_2 + \cdots + u_nu_{n-1})$$

$$\leq M^2 \quad (\text{bdd above})$$

$\Rightarrow \{S_n'\}$ partial sum of new series

H.W ① If $\sum u_n =$ conv series of +ve $\mathbb{R}$ then find conv/div. of following series:

$\sum u_n = \sum \frac{1}{n}$

(i) $\sum \sqrt{u_n}$ $\quad \sum u_n = \sum \frac{1}{n^2}$

(ii) $\sum \frac{\sqrt{u_n}}{\sqrt[4]{n}}$ H.W.

(iii) $\sum \frac{\sqrt{u_n}}{n}$ $\quad m = \frac{1}{2}$

(iv) $\sum \frac{\sqrt{u_n}}{\sqrt{n}}$

Hint: use $\sum \frac{1}{n^2}$ & AM-GM & power inequality

$$\left(\frac{u_1^m + \cdots + u_n^m}{n}\right) \le \left(\frac{u_1 + \cdots + u_n}{n}\right)^m$$

$m \in (0,1) \in \mathbb{Q}$, $\boxed{u_i > 0}$

$\frac{1}{n^2}$

$\downarrow$ reverse inequality for

$m \notin (0,1) \in \mathbb{Q}$

$\mathbb{Q}$ = Quotient set

rate of conv $u_n \simeq \frac{1}{n}$

Theorem (Pringsheim)

A necessary condition for convergence of an infinite series of positive real $\sum u_n$ with $\{u_n\} \downarrow$ (decreasing) is $\lim_{n\to\infty} n \cdot u_n = 0$

$$\boxed{\lim_{n\to\infty} u_n = 0}$$

proof hint [Cauchy]

Ex: Find whether following series conv/div

(i) $\sum \frac{1}{n+2}$ div

(ii) $\sum \frac{1}{n}$ div $\longrightarrow \lim \frac{n}{n} = 1 \neq 0$

(iii) $\sum 1/\sqrt{n}$ div

$\sum \frac{1}{\sqrt{n}}$ $\lim_{n\to\infty} n u_n = \frac{n}{\sqrt{n}} = \sqrt{n} = \infty$

V.V.V. Imp

overview chart

Convergence Tests
(for +ve $\mathbb{R}$ series)

- Comparison Test
- General form of Ratio & Root Test
- D'Alembert's Ratio Test
- Raabe's Test
- Gauss's Test
- Log test
- Cauchy Condensation Test

v. imp

overview chart

Convergence Tests for General series

(can have both +ve, -ve terms)

Alternating series → Leibnitz's test

Absolute & Conditional convergence

Abel's Test

Dirichlet's Test

Rearranged series → unconditionally convergent v.Imp

Conv. Test for +ve $\mathbb{R}$ series

Comparison Test

Let $\sum u_n$, $\sum v_n$ be two series of +ve $\mathbb{R}$

(A) $\boxed{\text{First Type :}}$ If $\exists$ a natural no. m ($m \in \mathbb{N}$ s.t) $u_n \le k v_n \ \forall \ n \ge m$ for a fixed no. k.

Then i) $\sum u_n$ conv if $\sum v_n$ conv
ii) $\sum u_n$ div if $\sum v_n$ div

ie $\sum u_n$, $\sum v_n$ conv/div together

v.Imp

(B) $\boxed{\text{Limit form :}}$

If $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ $(0 < l < \infty)$ then $\sum u_n, \sum v_n$ conv/div together

(C) $\boxed{\text{Second Type :}}$ If $\exists$ $m \in \mathbb{N}$ s.t. $\frac{u_{n+1}}{u_n} \le \frac{v_{n+1}}{v_n}$ $\forall n \ge m$, Then $\sum u_n, \sum v_n$ conv/div together.

Examples

Find whether the following series conv/div

$\frac{n(n+1)}{2}$

(1) $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \dots \infty$

Ans: here $u_n = \frac{1+2+3+\dots+n+1}{(n+1)^3} = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{n+2}{2(n+1)^2}$

Consider $v_n = \frac{1}{n}$. Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{n(n+2)}{2(n+1)^2} = \frac{1}{2}$ (finite & $\neq 0$)

$\Rightarrow u_n, v_n$ conv/div together. As v_n div $\Rightarrow u_n$ div (Ans)

(2) $\sum (\sqrt{n^4+1}) - (\sqrt{n^4-1})$

Ans: $u_n = \sqrt{n^4+1} - \sqrt{n^4-1} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$ $\simeq \frac{1}{n^2}$

Consider $v_n = \frac{1}{n^2}$. Then proceed as in previous ex (1)

(3) $\sum \frac{\sin(nx)}{n^2}$ $\quad \sum \frac{\sin nx}{n}$ $\quad (x > 0$ is constant$)$

Ans: $u_n = 0 \le \left|\frac{\sin(nx)}{n^2}\right| \le \frac{1}{n^2} \quad \forall n$. Now, $\sum \frac{1}{n^2}$ conv $\Rightarrow \sum \frac{\sin(nx)}{n^2}$ also conv.

H.W. Find whether the following series conv/div

① $\frac{1}{1 \cdot 2^2} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 4^2} + \cdots$ $\quad u_n \simeq \frac{1}{n}$

$\frac{\sin x}{x}$ $\quad y = \frac{1}{x}$, $yx = 1$

② $\frac{1}{2 \cdot 3 \cdot 4} + \frac{3}{3 \cdot 4 \cdot 5} + \frac{5}{4 \cdot 5 \cdot 6} + \cdots \quad 2n+1$

③ $\frac{5}{2 \cdot 2 \cdot 4} + \frac{7}{4 \cdot 3 \cdot 5} + \frac{9}{6 \cdot 4 \cdot 6} + \cdots$

$\sum \frac{1}{n^p}$ $\quad p > 1 \Rightarrow$ Conv

④ $\sum \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$ $\quad \sum \frac{1}{n^{3/2}}$

H.W. ~~Examples~~ Conv/ divg ; (C/D)

(1) $\frac{1}{1\cdot 2^2} + \frac{1}{2\cdot 3^2} + \frac{1}{3\cdot 4^2} + \cdots$ Ans : (C)

(2) show $\sum \frac{n^p}{(n+1)^{p+\alpha}}$ conv for $\alpha > 1$

$\sum \frac{1}{n^\alpha}$ conv for $\alpha > 1$

$\sum \frac{1}{n}$ $\alpha = 1$

(3) $\sum \sin\left(\frac{1}{n}\right)$ $\lim_{x\to 0} \frac{\sin x}{x} = 1$ Ans : (D)

$\sum \frac{1}{n^2}$ $\simeq$

(4) $\sum \frac{1}{n} \sin\left(\frac{1}{n}\right)$ $\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$ Ans (C)

(5) $\sum \frac{2^n+1}{3^n+2}$ $\simeq \frac{1}{a^n}$ where $a > 1$, conv $\sum \left(\frac{2}{3}\right)^n$ Ans: (C)

(6) $\sum \left(\sqrt[3]{n^3+1} - n\right)$ Ans (C)

(7) $\sum \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$ Ans: (C)

(8) $\sum \frac{1}{n \log n}$ H.W. Ans: (D)

$\frac{1}{n \cdot \log n}$ $\frac{1}{n \log n}$ $\simeq \frac{1}{\log n}$ no conclu^n. 0

$\lim \frac{\frac{1}{n}}{\frac{n}{\log}}$

(9) $\sum \frac{3^n}{2^n + 3^n}$ Ans: (D)

(10) show that, if $\sum u_n$ conv series of +ve $\mathbb{R}$, then $\sum \frac{u_n}{n}$ is also conv. First, $\sum u_n^2$ conv as $\sum u_n$?

Ans:

$\frac{u_n}{n} < \frac{u_n^2 + \frac{1}{n^2}}{2}$ $\Rightarrow \sum \frac{u_n}{n} < \left(\frac{1}{2} \sum u_n^2\right) + \left(\frac{1}{2} \sum \frac{1}{n^2}\right)$

(GM < QM)